

GLOBAL REGULAR SOLUTIONS FOR THE 3D ZAKHAROV-KUZNETSOV EQUATION POSED ON A BOUNDED DOMAIN

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ABSTRACT. An initial-boundary value problem for the 3D Zakharov-Kuznetsov equation posed on bounded domains is considered. Existence and uniqueness of a global regular solution as well as exponential decay of the H^2 -norm for small initial data are proven.

1. INTRODUCTION

We are concerned with the existence, uniqueness and exponential decay of the H^2 -norm for global regular solutions to an initial-boundary value problem (IBVP) for the 3D Zakharov-Kuznetsov (ZK) equation

$$u_t + (c_s + u)u_x + u_{xxx} + u_{xyy} + u_{xzz} = 0 \quad (1.1)$$

which describes the propagation of nonlinear ionic-sonic waves in a plasma submitted to a magnetic field directed along the x axis and c_s is a positive constant corresponding to the sound velocity [28, 29, 32]. This equation is a three-dimensional analog of the well-known Korteweg-de Vries (KdV) equation

$$u_t + uu_x + u_{xxx} = 0. \quad (1.2)$$

Equations (1.1) and (1.2) are typical examples of so-called dispersive equations which attract considerable attention of both pure and applied mathematicians in the past decades. The KdV equation is probably most studied in this context. The theory of the initial-value problem (IVP henceforth) for (1.2) is considerably advanced today [1, 4, 12, 13, 27, 30].

Recently, due to physics and numerics needs, publications on initial-boundary value problems both in bounded and unbounded domains for dispersive equations have appeared [2, 3, 5, 6, 17, 18, 24, 31, 33]. In particular, it has been discovered that the KdV equation posed on a bounded interval possesses an implicit internal dissipation. This allowed to prove the exponential decay rate of small solutions for (1.2)

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posed on bounded intervals without adding any artificial damping term [17]. Similar results were proved for a wide class of dispersive equations of any odd order with one space variable [10].

However, (1.2) is a satisfactory approximation for real waves phenomena while the equation is posed on the whole line ($x \in \mathbb{R}$); if cutting-off domains are taken into account, (1.2) is no longer expected to mirror an accurate rendition of reality. The correct equation in this case (see, for instance, [2, 33]) should be written as

$$u_t + u_x + uu_x + u_{xxx} = 0. \quad (1.3)$$

Indeed, if $x \in \mathbb{R}$, $t > 0$, the linear traveling term u_x in (1.3) can be easily scaled out by a simple change of variables, but it can not be safely ignored for problems posed both on finite and semi-infinite intervals without changes in the original domain.

Once bounded domains are considered as a spatial region of waves propagation, their sizes appear to be restricted by certain critical conditions. We recall, however, that if the transport term u_x is neglected, then (1.3) becomes (1.2), and it is possible to prove the exponential decay rate of small solutions for (1.2) posed on any bounded interval. More results on control and stabilizability for the KdV equation can be found in [25, 26].

Later, the interest on dispersive equations became to be extended for the multi-dimensional models such as Kadomtsev-Petviashvili (KP) and ZK equations. As far as the ZK equation is concerned, results both on IVP and IBVP can be found in [8, 9, 11, 20, 21, 22, 23, 28]. The biggest part of these publications is devoted to study of well-posedness of the Cauchy problem and initial-boundary value problems for the 2D ZK equation [8, 9, 11, 20, 21]. In the case of the 3D ZK equation, there are results on local well-posedness for the Cauchy problem [22, 23]; the existence of local strong solutions to an initial-boundary value problem posed on a bounded domain, [31], as well as the existence of global weak solutions [28].

Our work has been inspired by [28, 31] where (1.1) posed on a bounded domain was considered. A thorough analysis of these papers has revealed that an implicit dissipativity of the terms $u_{xyy} + u_{xzz}$ may help to establish a global well-posedness of initial-boundary value problems in classes of regular solutions. Yearlier this dissipativity has been used in order to prove exponential decay for the 2D ZK equation [16, 19].

The main goal of our work is to prove the existence and uniqueness of global-in-time regular solutions of (1.1) posed on bounded domains and the exponential decay rate of these solutions for sufficiently small initial

data. To cope with this problem, we exploited the strategy completely different from the standard schemes: first to prove the existence result and after that to study uniqueness and decay properties of solutions. In our case, we prove simultaneously existence of global regular solutions and their exponential decay.

The paper is outlined as follows. Section I is Introduction. Section 2 contains formulation of the problem and auxiliaries. In Section 3, we prove the existence and uniqueness of global regular solutions and, simultaneously, exponential decay of the H^2 -norm establishing global estimates of local strong solutions provided by [31].

2. PROBLEM AND PRELIMINARIES

Let L, B_y, B_z be finite positive numbers. Define

$$\begin{aligned}\mathcal{D} &= \{(x, y, z) \in \mathbb{R}^3 : x \in (0, L), y \in (0, B_y), z \in (0, B_z)\}; \\ \mathcal{S} &= \{(y, z) \in \mathbb{R}^2 : y \in (0, B_y), z \in (0, B_z)\}, \quad \mathcal{Q}_t = \mathcal{D} \times (0, t).\end{aligned}$$

Consider the following IBVP:

$$Au \equiv u_t + (c_s + u)u_x + \Delta u_x = 0 \quad \text{in } \mathcal{Q}_t; \quad (2.1)$$

$$u|_\gamma = 0, \quad t > 0; \quad (2.2)$$

$$u_x(L, y, z, t) = 0, \quad y \in (0, B_y), \quad z \in (0, B_z), \quad t > 0; \quad (2.3)$$

$$u(x, y, z, 0) = u_0(x, y, z), \quad (x, y, z) \in \mathcal{D}, \quad (2.4)$$

where γ denotes the boundary of \mathcal{D} , $u_0 : \mathcal{D} \rightarrow \mathbb{R}$ is a given function.

Hereafter subscripts u_x , u_{xy} , etc. denote the partial derivatives, as well as ∂_x or ∂_{xy}^2 when it is convenient. Operators ∇ and Δ are the gradient and Laplacian acting over \mathcal{D} . By (\cdot, \cdot) and $\|\cdot\|$ we denote the inner product and the norm in $L^2(\mathcal{D})$, and $\|\cdot\|_{H^k}$ stands for the norm in L^2 -based Sobolev spaces.

We will need the following result [14].

Lemma 2.1. *Let $u \in H^1(\mathcal{D})$ and γ be the boundary of \mathcal{D} .*

If $u|_\gamma = 0$, then

$$\|u\|_{L^q(\mathcal{D})} \leq 4^\theta \|\nabla u\|^\theta \|u\|^{1-\theta}, \quad (2.5)$$

where $\theta = 3\left(\frac{1}{2} - \frac{1}{q}\right)$.

If $u|_\gamma \neq 0$, then

$$\|u\|_{L^q(\mathcal{D})} \leq 4^\theta C_{\mathcal{D}} \|u\|_{H^1(\mathcal{D})}^\theta \|u\|^{1-\theta}, \quad (2.6)$$

where $C_{\mathcal{D}}$ does not depend on a size of \mathcal{D} .

3. EXISTENCE THEOREM

In this section we state the existence result for bounded domains.

Theorem 3.1. *Let u_0 be a given function such that $u_0|_\gamma = u_{0x}|_{x=L} = 0$ and*

$$\|u_0\|^2 + \|u_{0yy}\|^2 + \|u_{0zz}\|^2 + J_0 < \infty,$$

where

$$J_0 = ((1+x), u_0^2 + [(c_s + u_0)u_{0x} + \Delta u_{0x}]^2).$$

Moreover, the following conditions to be fulfilled:

$$\begin{aligned} K_2 &= \pi^2 \left[\frac{7}{8B_y^2} + \frac{7}{8B_z^2} + \frac{23}{8L^2} \right] \geq 4c_s; \\ \|u_0\|^4 &\leq \frac{K_2}{4K_3}; \quad J_0^2 \leq \frac{K_2}{4K_4}, \end{aligned} \quad (3.1)$$

where

$$K_3 = 3^3 2^{16} (1+L)^4 (2C_1^2 + 1), \quad C_1 = 1 + c_s + \frac{2^{11}}{3} \|u_0\|^4, \quad K_4 = \frac{3^3 2^{19}}{25} (1+L)^6.$$

Then there exists a unique regular solution to (2.1)-(2.4) such that

$$\begin{aligned} u &\in L^\infty(0, \infty; H^2(\mathcal{D})) \cap L^2(0, \infty; H^3(\mathcal{D})); \\ \Delta u_x &\in L^\infty(0, \infty; L^2(\mathcal{D})) \cap L^2(0, \infty; H^1(\mathcal{D})); \\ u_t &\in L^\infty(0, \infty; L^2(\mathcal{D})) \cap L^2(0, \infty; H^1(\mathcal{D})) \end{aligned}$$

and

$$\|u\|_{H^2(\mathcal{D})}^2(t) \leq C e^{-\chi t} J_0. \quad (3.2)$$

where the constant C depends on L, J_0 ; $\chi = \frac{K_2}{4(1+L)}$.

Proof. To prove this theorem, we use local in t existence of strong solutions to (2.1)-(2.4) established in [31] and prove global a priori estimates of strong solutions. Of course, it is possible to use a parabolic regularization as in [19, 31] and to prove directly global estimates of regular solutions for a parabolic problem.

3.1. Estimate I. Multiply (2.1) by u and integrate over \mathcal{D} and $(0, t)$ to obtain

$$\|u\|^2(t) + \int_0^t \int_S u_x^2(0, y, z, \tau) dy dz d\tau = \|u_0\|^2, \quad t \in (0, T). \quad (3.3)$$

The following inequalities are crucial for our proof.

Proposition 3.1. *Let $v \in H_0^1(D)$. Then*

$$\begin{aligned} \|v_y\|^2(t) &\geq \frac{\pi^2}{B_y^2} \|v\|^2(t), \quad \|v_z\|^2(t) \geq \frac{\pi^2}{B_z^2} \|v\|^2(t), \\ \|v_x\|^2(t) &\geq \frac{\pi^2}{L^2} \|v\|^2(t). \end{aligned} \quad (3.4)$$

Proof. The proof is based on the Steklov inequality: let $v(t) \in H_0^1(0, \pi)$, then $\int_0^\pi v_t^2(t) dt \geq \int_0^\pi v^2(t) dt$. Inequalities (3.4) follow from here by a simple scaling. \square

3.2. Estimate II. Write the inner product

$$2(Au, (1+x)u)(t) = 0$$

as

$$\begin{aligned} \frac{d}{dt}((1+x), u^2)(t) + \int_S u_x^2(0, y, z, t) dydz - c_s \|u\|^2(t) \\ + 3\|u_x\|^2(t) + \|u_y\|^2(t) + \|u_z\|^2(t) = \frac{2}{3}(1, u^3)(t). \end{aligned} \quad (3.5)$$

Making use of (2.5), we compute

$$\begin{aligned} I = \frac{2}{3}(1, u^3)(t) &\leq \frac{2}{3} \|u\|_{L^3(D)}^3(t) \leq \frac{2^4}{3} [\|\nabla u\|^{1/2}(t) \|u\|^{1/2}(t)]^3 \\ &\leq \frac{1}{8} \|\nabla u\|^2(t) + \frac{2^{17}}{3} \|u\|^6(t). \end{aligned} \quad (3.6)$$

Substituting I into (3.5), we obtain

$$\begin{aligned} \frac{d}{dt}((1+x), u^2)(t) + \int_S u_x^2(0, y, z, t) dydz \\ + \frac{23}{8} \|u_x\|^2(t) + \frac{7}{8} \|u_y\|^2(t) + \frac{7}{8} \|u_z\|^2(t) \\ - K_1 \|u\|^6(t) - c_s \|u\|^2(t) \leq 0, \end{aligned}$$

where $K_1 = \frac{2^{17}}{3}$.

Using (3.4), we get

$$\begin{aligned} \frac{d}{dt}((1+x), u^2)(t) + \frac{K_2}{2} \|u\|^2(t) + \int_S u_x^2(0, y, z, t) dydz \\ \left[\frac{K_2}{2} - c_s - K_1 \|u\|^4(t) \right] \|u\|^2(t), \end{aligned} \quad (3.7)$$

where

$$K_1 = \frac{2^{17}}{3}.$$

By conditions of Theorem 3.1, the last inequality can be rewritten as

$$\frac{d}{dt} ((1+x), u^2)(t) + 2\chi((1+x), u^2)(t) \leq 0,$$

with $\chi = \frac{K_2}{4(1+L)}$. Solving this inequality, we find

$$\|u\|^2(t) \leq ((1+x), u^2)(t) \leq e^{-2\chi t}((1+x), u_0^2) \quad \forall t > 0. \quad (3.8)$$

3.3. Estimate III. Rewrite the scalar product

$$2(Au, (1+x)u)(t) = 0$$

as

$$\begin{aligned} & \int_S u_x^2(0, y, z, t) dydz + 3\|u_x\|^2(t) + \|u_y\|^2(t) + \|u_z\|^2(t) \\ &= \frac{2}{3}(1, u^3)(t) + c_s\|u\|^2(t) - 2((1+x)u, u_t)(t). \end{aligned}$$

Using (2.5), we find

$$I = \frac{2}{3}(1, u^3)(t) \leq \frac{1}{2}\|\nabla u\|^2(t) + \frac{2^{11}}{3}\|u\|^6(t).$$

Substituting I into the last equation, we get

$$\begin{aligned} & \int_S u_x^2(0, y, z, t) dydz + \frac{5}{2}\|u_x\|^2(t) + \frac{1}{2}\|u_y\|^2(t) + \frac{1}{2}\|u_z\|^2(t) \\ & \leq c_s\|u\|^2(t) + \frac{2^{11}}{3}\|u\|^6(t) - 2((1+x)u, u_t)(t) \end{aligned} \quad (3.9)$$

and

$$\begin{aligned} \|u_x\|^2(t) & \leq \frac{2}{5}\|u\|^2(t) \left[1 + c_s + \frac{2^{11}}{3}\|u\|^4(t) \right] + \frac{2}{5}(1+L)((1+x), u_t^2)(t) \\ & \leq C_1\|u\|^2(t) + \frac{2}{5}(1+L)((1+x), u_t^2)(t), \end{aligned} \quad (3.10)$$

where

$$C_1 = \frac{2}{5}(1 + c_s + \frac{2^{11}}{3}\|u_0\|^4).$$

3.4. **Estimate IV.** Write the inner product

$$((Au)_t, (1+x)u_t)(t) = 0$$

as

$$\begin{aligned} \frac{d}{dt} ((1+x), u_t^2)(t) + \int_{\mathcal{S}} u_{xt}^2(0, y, z, t) dydz - c_s \|u_t\|^2(t) + 3\|u_{xt}\|^2(t) \\ + \|u_{yt}\|^2(t) + \|u_{zt}\|^2(t) + 2((1+x)(uu_x)_t, u_t)(t) = 0. \end{aligned} \quad (3.11)$$

We calculate

$$\begin{aligned} I &= 2((1+x)(uu_x)_t, u_t)(t) = 2((1+x)(uu_t)_x, u_t)(t) \\ &= ((1+x)u_x - u, u_t^2)(t) \leq \|(1+x)u_x - u\|(t) \|u_t\|_{L^4(\mathcal{D})}^2(t) \\ &\leq (1+L) [\|u_x\|(t) + \|u\|(t)] 4^{3/2} \|u_t\|^{1/2}(t) \|\nabla u_t\|^{3/2}(t) \\ &\leq \frac{3}{4} \epsilon^{4/3} \|\nabla u_t\|^2(t) + \frac{1}{4\epsilon^4} 4^6 (1+L)^4 [\|u_x\|(t) + \|u\|(t)]^4 \|u_t\|^2(t). \end{aligned}$$

Taking $\frac{3}{4}\epsilon^{4/3} = \frac{1}{8}$, we get

$$I \leq \frac{1}{8} \|\nabla u_t\|^2(t) + (1+L)^4 3^3 2^{16} [\|u_x\|^4(t) + \|u\|^4(t)] \|u_t\|^2(t).$$

Substituting I into (3.11) and making use of (3.10), we obtain

$$\begin{aligned} \frac{d}{dt} ((1+x), u_t^2)(t) + \int_{\mathcal{S}} u_{xt}^2(0, y, z, t) dydz + \frac{1}{4} \left\{ \frac{23}{8} \|u_{xt}\|^2(t) \right. \\ \left. + \frac{7}{8} \|u_{yt}\|^2(t) + \frac{7}{8} \|u_{zt}\|^2(t) \right\} + \frac{3}{4} K_2 - \{c_s + (1+L)^4 3^3 2^{16} (2C_1^2 + 1) \|u_0\|^4 \\ + \frac{3^3 2^{19}}{25} (1+L)^6 ((1+x), u_t^2)^2(t)\} \|u_t\|^2(t) \leq 0 \end{aligned}$$

which can be rewritten as

$$\begin{aligned} \frac{d}{dt} ((1+x), u_t^2)(t) + \int_{\mathcal{S}} u_{xt}^2(0, y, z, t) dydz + \frac{1}{4} \left\{ \frac{23}{8} \|u_{xt}\|^2(t) \right. \\ \left. + \frac{7}{8} \|u_{yt}\|^2(t) + \frac{7}{8} \|u_{zt}\|^2(t) \right\} + \left[\frac{3K_2}{4} - c_s - K_3 \|u_0\|^4 \right. \\ \left. - K_4 ((1+x), u_t^2)^2(t) \right] \|u_t\|^2(t) \leq 0, \end{aligned} \quad (3.12)$$

where

$$K_3 = 3^3 2^{16} (1+L)^4 (2C_1^2 + 1), \quad K_4 = \frac{3^3 2^{19} (1+L)^6}{25}. \quad (3.13)$$

Due to conditions of Theorem 3.1,

$$K_3 \|u_0\|^4 < \frac{K_2}{4}, \quad K_4 ((1+x), u_t^2)^2(0) < \frac{K_2}{4},$$

hence, [10],

$$K_4((1+x), u_t^2)^2(t) < \frac{K_2}{4} \quad \forall t > 0$$

and (3.12) becomes

$$\begin{aligned} & \frac{d}{dt} ((1+x), u_t^2)(t) + \frac{1}{4} \left\{ \frac{23}{8} \|u_{xt}\|^2(t) + \frac{7}{8} \|u_{yt}\|^2(t) \right. \\ & \left. + \frac{7}{8} \|u_{zt}\|^2(t) \right\} \leq 0. \end{aligned}$$

Making use of (3.4), we get

$$\frac{d}{dt} ((1+x), u_t^2)(t) + \frac{K_2}{4(1+L)} ((1+x), u_t^2)(t) \leq 0.$$

Since

$$((1+x), u_t^2)(0) = ((1+x), [(c_s + u_0)u_{0x} + \Delta u_{0x}]^2) \leq J_0,$$

solving this inequality, we find

$$\|u_t\|^2 \leq ((1+x), u_t^2)(t) \leq e^{-\chi t} ((1+x), u_t^2)(0) \leq e^{-\chi t} J_0 \quad (3.14)$$

with $\chi = \frac{K_2}{4(1+L)}$.

Returning to (3.12) and taking into account (3.8), (3.9), we obtain

$$\begin{aligned} & ((1+x), u_t^2)(t) + \int_S u_x^2(0, y, z, t) dydz + \|u\|_{H_0^1(\mathcal{D})}^2(t) \\ & + \int_S u_x^2(0, y, z, t) dydz \leq C_2 e^{-\chi t} J_0; \end{aligned} \quad (3.15)$$

$$\begin{aligned} & \int_0^t \left\{ \int_S u_x^2(0, y, z, \tau) dydz + \|\nabla u_\tau\|^2(\tau) \right\} d\tau \\ & \leq C_3 J_0 \quad \forall t > 0, \end{aligned} \quad (3.16)$$

where the constants C_2, C_3 do not depend on $t > 0$.

3.5. Estimate V. Transform the scalar product

$$-2((1+x)Au, u_{yy} + u_{zz})(t) = 0$$

into the following equality:

$$\begin{aligned} & -c_s (\|u_y\|^2(t) + \|u_z\|^2(t)) + \int_S [u_{xy}^2(0, y, z, t) + u_{xz}^2(0, y, z, t)] dydz \\ & + \|u_{yy}\|^2(t) + \|u_{zz}\|^2(t) + 2\|u_{yz}\|^2(t) + 3\|u_{xy}\|^2(t) + 3\|u_{xz}\|^2(t) \\ & + ((1+x)u_x - u, u_y^2)(t) + ((1+x)u_x - u, u_z^2)(t) \\ & = 2((1+x)u_t, u_{yy} + u_{zz})(t). \end{aligned} \quad (3.17)$$

We estimate

$$\begin{aligned}
I_1 &= ((1+x)u_x - u, u_y^2)(t) \leq \|(+x)u_x - u\|(t) \|u_y\|_{L^4(\mathcal{D})}^2(t) \\
&\leq (1+L) [\|u_x\|(t) + \|u\|(t)] 4^{3/2} C_D^2 \|\nabla u\|^{1/2}(t) \|\nabla u_y\|^{3/2}(t) \\
&\leq \frac{1}{8} \|\nabla u_y\|^2(t) + (1+L)^4 C_D^8 2^{16} 3^3 [\|u_x\|^4(t) + \|u\|^4(t)] \|\nabla u\|^2(t).
\end{aligned}$$

Similarly,

$$\begin{aligned}
I_2 &= ((1+x)u_x - u, u_z^2)(t) \leq \frac{1}{8} \|\nabla u_z\|^2(t) \\
&\quad + (1+L)^4 C_D^8 2^{16} 3^3 [\|u_x\|^4(t) + \|u\|^4(t)] \|\nabla u\|^2(t).
\end{aligned}$$

Substituting I_1, I_2 into (3.17), we find

$$\begin{aligned}
&\int_{\mathcal{S}} [u_{xy}^2(0, y, z, t) + u_{xz}^2(0, y, z, t)] dydz + \|u_{yy}\|^2(t) + \|u_{zz}\|^2(t) \\
&\quad + \|u_{yz}\|^2(t) + \|u_{xy}\|^2(t) + \|u_{xz}\|^2(t) \\
&\leq C_4(L) [\|\nabla u\|^6(t) + \|u\|^4(t) \|\nabla u\|^2(t) + ((1+x), u_t^2)(t)].
\end{aligned}$$

Making use of (3.8), (3.9), (3.14), (3.15), we get

$$\begin{aligned}
&\int_{\mathcal{S}} [u_{xy}^2(0, y, z, t) + u_{xz}^2(0, y, z, t)] dydz + \|u_y\|_{H^1(\mathcal{D})}^2(t) \\
&\quad + \|u_z\|_{H^1(\mathcal{D})}^2(t) \leq C_5(L, J_0) e^{-\chi t} J_0.
\end{aligned} \tag{3.18}$$

To prove that

$$\|u\|_{H^2(\mathcal{D})}^2(t) \leq C_6(L, J_0) e^{-\chi t} J_0,$$

it is necessary to estimate $\|u_{xx}\|(t)$.

3.6. Estimate VI. Strong solutions to (2.1)-(2.4) satisfy the following elliptic problem:

$$\Delta u_x = -u_t - c_s u_x - \frac{1}{2} (u^2)_x; \tag{3.19}$$

$$u_x(0, y, z, t) = \phi(y, z, t); \tag{3.20}$$

$$u_x(L, y, z, t) = u_x(x, 0, z, t) = u_x(x, B_y, z, t) \tag{3.21}$$

$$= u_x(x, y, 0, t) = u_x(x, y, B_z, t) = 0. \tag{3.22}$$

Denote $v = u_x - \phi(y, z)(1 - \frac{x}{L})$ to come to the following Dirichlet problem:

$$\begin{aligned}\Delta v &= -u_t - c_s u_x - \frac{1}{2}(u^2)_x - \left(1 - \frac{x}{L}\right)(\phi_y(y, z, t))_y \\ &\quad - \left(1 - \frac{x}{L}\right)(\phi_z(y, z, t))_z \equiv F(x, y, z, t); \end{aligned} \quad (3.23)$$

$$v|_\gamma = 0. \quad (3.24)$$

Considering the scalar product

$$-(\Delta v, v)(t) = -(F, v)(t),$$

we find

$$\begin{aligned}\|v_x\|^2(t) + \|v_y\|^2(t) + \|v_z\|^2(t) &\leq C_7 \{ \|u_t\|^2(t) + \|u\|^2(t) + \|u\|_{L^4(\mathcal{D})}^4(t) \\ &\quad + \int_{\mathcal{S}} [(u_{xy}^2(0, y, z, t) + u_{xz}^2(0, y, z, t))] dydz \}. \end{aligned} \quad (3.25)$$

We estimate

$$\|u\|_{L^4(\mathcal{D})}^4(t) \leq 4^3 \|u\|(t) \|\nabla u\|^3(t).$$

Substituting this into (3.25) and making use of (3.14), (3.15), we get

$$\|v_x\|^2(t) \leq C_8 e^{-\chi t} J_0.$$

By definition,

$$u_{xx} = v_x - \frac{1}{L} \phi(y, z, t).$$

Hence, due to (3.18),

$$\|u_{xx}\|^2(t) \leq C_9 e^{-\chi t} J_0$$

which jointly with (3.18) reads

$$\|u\|_{H^2(\mathcal{D})}^2(t) \leq C_{10}(L, J_0) e^{-\chi t}. \quad (3.26)$$

3.7. Estimate VII. Consider the scalar product

$$2((1+x)Au, \partial_y^4 u + \partial_z^4 u + \partial_y^2 \partial_z^2 u)(t) = 0$$

and transform it into the equality

$$\begin{aligned}
& \frac{d}{dt}((1+x), u_{yy}^2 + u_{zz}^2 + u_{yz}^2)(t) \\
& - c_s [\|u_{yy}\|^2(t) + \|u_{zz}\|^2(t) + \|u_{yz}\|^2(t)] \\
& + \int_{\mathcal{S}} \{u_{xyy}^2(0, y, z, t) + u_{xzz}^2(0, y, z, t) + u_{xyz}^2(0, y, z, t)\} dydz \\
& + \|u_{yyy}\|^2(t) + \|u_{zzz}\|^2(t) + 2\|u_{yzz}\|^2(t) + 2\|u_{zyy}\|^2(t) \\
& + 3\|u_{xyy}\|^2(t) + 3\|u_{xzz}\|^2(t) + 3\|u_{xyz}\|^2(t) \\
& + 2((1+x)uu_x, \partial_y^4 u)(t) + 2((1+x)uu_x, \partial_z^4 u)(t) \\
& + 2((1+x)uu_x, \partial_y^2 \partial_z^2 u)(t) = 0.
\end{aligned} \tag{3.27}$$

We estimate

$$\begin{aligned}
I_1 &= 2((1+x)uu_x, \partial_z^4 u)(t) = -2((1+x)u_z u_x, \partial_z^3 u)(t) \\
& - 2((1+x)uu_{zx}, \partial_z^3 u)(t) = ((1+x)u_x - u, u_{zz}^2)(t) \\
& - 2(u_z^2, u_{zz})(t) - 2((1+x)u_z^2, u_{xzz})(t) \equiv I_{11} + I_{12} + I_{13},
\end{aligned}$$

where

$$\begin{aligned}
I_{11} &= ((1+x)u_x - u, u_{zz}^2)(t) \leq \|(1+x)u_x - u\|(t) \|u_{zz}\|_{L^4(\mathcal{D})}^2(t) \\
& \leq (1+L)[\|u_x\|(t) + \|u\|(t)] 4^{3/2} \|u_{zz}\|^{1/2}(t) \|\nabla u_{zz}\|^{3/2}(t) \\
& \leq \frac{3}{4} \epsilon^{4/3} \|\nabla u_{zz}\|^2(t) + \frac{(1+L)^4}{\epsilon^4} 2^{13} \|u_{zz}\|^2(t) [\|u_x\|^4 + \|u\|^4];
\end{aligned}$$

$$I_{12} = 2(u_z^2, u_{zz})(t) \leq 2^4 \|u_{zz}\|(t) \|u_z\|^{1/2}(t) \|\nabla u_z\|^{3/2}(t) \leq C \|u\|_{H^2(\mathcal{D})}^3(t)$$

and

$$\begin{aligned}
I_{13} &= 2((1+x)u_z^2, u_{xzz})(t) \leq 4^3 (1+L) \|u_{xzz}\|(t) \|u_z\|^{1/2}(t) \|\nabla u_z\|^{3/2}(t) \\
& \leq \delta_1 \|u_{xzz}\|^2(t) + \frac{4^3 (1+L)^2}{\delta_1} \|u_z\|(t) \|\nabla u_z\|^3(t).
\end{aligned}$$

Similarly,

$$\begin{aligned}
I_2 &= 2((1+x)uu_x, \partial_y^4 u)(t) = -2((1+x)u_y u_x, \partial_y^3 u)(t) \\
& - 2((1+x)uu_{yx}, \partial_y^3 u)(t) = ((1+x)u_x - u, u_{yy}^2)(t) \\
& - 2(u_y^2, u_{yy})(t) - 2((1+x)u_y^2, u_{xyy})(t) \equiv I_{21} + I_{22} + I_{23},
\end{aligned}$$

where

$$\begin{aligned}
I_{21} &= ((1+x)u_x - u, u_{yy}^2)(t) \leq \|(1+x)u_x - u\|(t) \|u_{yy}\|_{L^4(\mathcal{D})}^2(t) \\
&\leq \frac{3}{4}\epsilon^{4/3} \|\nabla u_{yy}\|^2(t) + \frac{(1+L)^4}{\epsilon^4} 2^{13} \|u_{yy}\|^2(t) [\|u_x\|^4(t) + \|u\|^4(t)];
\end{aligned}$$

$$I_{22} = 2(u_y^2, u_{yy})(t) \leq 2^4 \|u_{yy}\|(t) \|u_y\|^{1/2}(t) \|\nabla u_y\|^{3/2}(t) \leq 2^4 \|u\|_{H^2(\mathcal{D})}^3(t)$$

and

$$I_{23} = 2((1+x)u_y^2, u_{xyy})(t) \leq \delta_1 \|u_{xyy}\|^2(t) + \frac{4^3(1+L)^2}{\delta_1} \|u_y\|(t) \|\nabla u_y\|^3(t).$$

$$\begin{aligned}
I_3 &= 2((1+x)uu_x, \partial_y^2 \partial_z^2 u)(t) = -2((1+x)u_z u_x, \partial_y^2 \partial_z u)(t) \\
&\quad - 2((1+x)uu_{zx}, \partial_y^2 \partial_z u)(t) = 2((1+x)u_x, u_{zy}^2)(t) \\
&\quad - 2((1+x)u_z^2, u_{xyy})(t) - (u_z^2, u_{yy})(t) + 2((1+x)uu_{xzz}, u_{yy})(t) \\
&\equiv I_{31} + I_{32} + I_{33} + I_{34}.
\end{aligned}$$

From here

$$\begin{aligned}
I_{31} &= 2((1+x)u_x, u_{zy}^2)(t) \leq 2(1+L) \|u_x\|(t) \|u_{zy}\|_{L^4(\mathcal{D})}^2(t) \\
&\leq 2^4(1+L) \|u_x\|(t) \|u_{zy}\|^{1/2}(t) \|\nabla u_{zy}\|^{3/2}(t) \\
&\leq \frac{3}{4}\epsilon^{4/3} \|\nabla u_{zy}\|^2(t) + \frac{2^{14}}{\epsilon^4} (1+L)^4 \|u_x\|^4(t) \|u_{zy}\|^2(t);
\end{aligned}$$

$$\begin{aligned}
I_{32} &= -2((1+x)u_z^2, u_{xyy})(t) \leq (1+L) \|u_{xyy}\|(t) \|u_z\|_{L^4(\mathcal{D})}^2(t) \\
&\leq \delta \|u_{xyy}\|^2(t) + \frac{4^3(1+L)^2}{\delta} \|\nabla u\|(t) \|\nabla u_z\|^3(t);
\end{aligned}$$

$$\begin{aligned}
I_{33} &= (u_z^2, u_{yy})(t) \leq \|u_{yy}\|(t) \|u_z\|_{L^4(\mathcal{D})}^2(t) \\
&\leq \|u_{yy}\|^2(t) + 4^2 \|u_z\|(t) \|\nabla u_z\|^3(t);
\end{aligned}$$

$$\begin{aligned}
I_{34} &= 2((1+x)uu_{xzz}, u_{yy})(t) \leq 2(1+L) \|u_{xzz}\|(t) \|u\|_{L^4(\mathcal{D})}(t) \|u_{yy}\|_{L^4(\mathcal{D})}(t) \\
&\leq \delta_1 \|u_{xzz}\|^2(t) + \frac{4^3(1+L)^2}{\delta_1} \|\nabla u_{yy}\|^{3/2}(t) \|u\|^{1/2}(t) \|\nabla u\|^{3/2}(t) \|u_{yy}\|^{1/2}(t) \\
&\leq \delta_1 \|u_{xzz}\|^2(t) + \frac{3\delta^{4/3}}{4\delta_1} \|\nabla u_{yy}\|^2(t) + \frac{4^{11}(1+L)^8}{\delta_1 \delta^4} \|u\|^2(t) \|u_{yy}\|^2(t) \|\nabla u\|^6(t),
\end{aligned}$$

where $\delta, \delta_1, \epsilon$ are arbitrary positive numbers.

Taking them sufficiently small, we reduce (3.27) to the form

$$\begin{aligned}
& \frac{d}{dt}((1+x), u_{yy}^2 + u_{zz}^2 + u_{yz}^2)(t) \\
& + \int_{\mathcal{S}} \{u_{xyy}^2(0, y, z, t) + u_{xzz}^2(0, y, z, t) + u_{xyz}^2(0, y, z, t)\} dydz \\
& + \|u_{yyy}\|^2(t) + \|u_{zzz}\|^2(t) + \|u_{yzz}\|^2(t) + \|u_{zyy}\|^2(t) \\
& + \|u_{xyy}\|^2(t) + \|u_{xzz}\|^2(t) + \|u_{xyz}\|^2(t) \\
& \leq C_{11}(L) \|u\|_{H^2(\mathcal{D})}^3(t) [1 + \|u\|_{H^2(\mathcal{D})}^3(t)].
\end{aligned}$$

Integrating this, we obtain

$$\begin{aligned}
& ((1+x), u_{yy}^2 + u_{zz}^2 + u_{yz}^2)(t) \\
& + \int_0^t \left\{ \int_{\mathcal{S}} \{u_{xyy}^2(0, y, z, \tau) + u_{xzz}^2(0, y, z, \tau) + u_{xyz}^2(0, y, z, \tau)\} dydz \right. \\
& + \|u_{yyy}\|^2(\tau) + \|u_{zzz}\|^2(\tau) + \|u_{yzz}\|^2(\tau) + \|u_{zyy}\|^2(\tau) \\
& + \|u_{xyy}\|^2(\tau) + \|u_{xzz}\|^2(\tau) + \|u_{xyz}\|^2(\tau) \Big\} d\tau \\
& \leq C_{12}(L, B_y, B_z, J_0) ((1+x), u_{0yy}^2 + u_{0zz}^2 + u_{0zy}^2), \tag{3.28}
\end{aligned}$$

with the constant $C_{12}(L, B_y, B_z, J_0)$ independent of $t > 0$.

Lemma 3.1. *Strong solutions to (2.1)-(2.4) satisfy the following inclusions:*

$$u \in L^\infty(0, \infty; H^2(\mathcal{D})) \cap L^2(0, \infty; H^3(\mathcal{D})); \tag{3.29}$$

$$\Delta u_x \in L^\infty(0, \infty; L^2(\mathcal{D})) \cap L^2(0, \infty; H^1(\mathcal{D})); . \tag{3.30}$$

Proof. First, we will prove (3.29). For this purpose, rewrite (2.1) in the form

$$\begin{aligned}
\Delta v &= u_t - c_s u_x - u u_x - \left(1 - \frac{x}{L}\right) \Phi_{yy}(y, z, t) \\
& - \left(1 - \frac{x}{L}\right) \Phi_{zz}(y, z, t) \equiv F(x, y, z, t); \tag{3.31}
\end{aligned}$$

$$v|_\gamma = 0, \tag{3.32}$$

where

$$\Phi(y, z, t) \equiv u_x(0, y, z, t), \quad v = u_x - \Phi(y, z, t) \left(1 - \frac{x}{L}\right).$$

Due to (3.28),

$$F \in L^2(0, \infty; L^2(\mathcal{D})).$$

This implies that the Dirichlet problem (3.31), (3.32) has a unique solution

$$v \in L^2(0, \infty; H^2(\mathcal{D})),$$

hence,

$$u_x \in L^2(0, \infty; H^2(\mathcal{D})).$$

Taking into account (3.28), we prove (3.29). On the other hand, it follows from the equation

$$\Delta u_x = -u_t - c_s u_x - uu_x \equiv G(x, y, t)$$

that $G \in L^2(0, \infty; H^1(\mathcal{D}))$.

Indeed, making use of Proposition 2.5 and (3.29), we find

$$\begin{aligned} \|uu_x\|_{H^1(\mathcal{D})}(t) &\leq \|uu_x\|(t) + \|\nabla(uu_x)\|(t) \\ &\leq \|u\|_{L^4(\mathcal{D})}(t)\|u_x\|_{L^4(\mathcal{D})}(t) + \|u\|_{L^4(\mathcal{D})}(t)\|\nabla u_x\|_{L^4(\mathcal{D})}(t) + \|\nabla u\|_{L^4(\mathcal{D})}^2(t) \\ &\leq 2^3[\|u\|^{1/4}(t)\|\nabla u\|(t)\|\nabla u_x\|^{3/4}(t) \\ &\quad + \|u\|^{1/4}(t)\|\nabla u\|^{3/4}(t)\|\nabla u_x\|^{1/4}(t)\|\nabla^2 u_x\|^{3/4}(t) + \|\nabla u\|^{1/2}(t)\|\nabla^2 u\|^{3/2}(t)] \\ &\leq 2^3\|u\|_{H^3(\mathcal{D})}^2(t) \leq \infty. \end{aligned}$$

By (3.29), $uu_x \in L^2(0, \infty; H^1(\mathcal{D}))$. This implies that $G \in L^2(0, \infty; H^1(\mathcal{D}))$. Thus, the proof of Lemma 3.1 is complete. \square

The proof of the existence part of Theorem 3.1 is also complete.

Lemma 3.2. *The strong solution of Theorem 3.1 is uniquely defined.*

Proof. Let u_1, u_2 be distinct solutions to (2.1)-(2.4), then $w = u_1 - u_2$ satisfies the following initial-boundary value problem:

$$Lw \equiv w_t + c_s w_x + \Delta w_x + ww_x + (u_2 w)_x = 0; \text{ in } \mathcal{Q}_t; \quad (3.33)$$

$$w|_\gamma = w_x(L, y, z, t) = 0, \quad t > 0, (y, z) \in \mathcal{S}; \quad (3.34)$$

$$w(x, y, x, 0) \equiv 0 \quad (x, y, z) \in \mathcal{D}. \quad (3.35)$$

Consider the scalar product

$$2((1+x)Lw, w)(t) = 0$$

which can be transformed into the following equality:

$$\begin{aligned} \frac{d}{dt}((1+x), w^2)(t) + \int_{\mathcal{S}} w_x^2(0, y, z, t) dydz + 2\|w_x\|^2(t) + \|\nabla w\|^2(t) \\ - \frac{2}{3}(1, w^3)(t) + ((1+x)u_{2x} - u_2, w^2)(t) = 0. \end{aligned} \quad (3.36)$$

Making use of Proposition 2.5 and acting in the same manner as above, we find

$$I_1 = -\frac{2}{3}(1, w^3)(t) \leq \frac{3}{4}\epsilon^{4/3}\|\nabla w\|^2(t) + \frac{2^{14}}{3^4\epsilon^4}\|w\|^6(t)$$

and

$$\begin{aligned} I_2 &= 2((1+x)u_{2x} - u_2, w^2)(t) \leq \frac{3}{4}\epsilon^{4/3}\|\nabla w\|^2(t) \\ &+ \frac{C(L)}{\epsilon^4}[\|u_{2x}\|^4(t) + \|u_2\|^4(t)]\|w\|^2(t). \end{aligned}$$

Taking $\epsilon > 0$ sufficiently small, substituting I_1, I_2 into (3.36) and making use of (3.29), we obtain

$$\begin{aligned} \frac{d}{dt}((1+x), w^2)(t) &\leq C_{13}(L)[\|u_{2x}\|^4(t) \\ &+ \|u_1\|^4(t) + \|u_2\|^4(t)]((1+x), w^2)(t). \end{aligned}$$

By (3.18),

$$\|u_{2x}\|^4(t) + \|u_1\|^4(t) + \|u_2\|^4(t) \in L^1(0, \infty)$$

and by the Gronwall Lemma,

$$\|w\|^2(t) \leq ((1+x), w^2)(t) \equiv 0.$$

The proof of Lemma 3.2 is complete. \square

Obviously, this completes also the proof of Theorem 3.1. \square

Remark 3.1. *If $w(x, y, z, 0) = w_0(x, y, z) \neq 0$, then*

$$\|w\|^2(t) \leq ((1+x), w^2)(t) \leq C(L, J_0)((1+x), w_0^2) \quad \forall t > 0.$$

This means continuous dependence of solutions to (2.1)-(2.4) on initial data.

Remark 3.2. *To prove Theorem 1, we have the following alternative: either the coefficient K_2 to be sufficiently large which can be done for small values of L, B_y, B_z (at least one of them) or initial data $\|u_0\|, J_0$ to be sufficiently small while c_s is fixed. In our case, L may be arbitrary large finite, hence we can handle values of B_y, B_z .*

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